

# The Complete Bifurcation Diagram for the Logistic Map

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The complete bifurcation diagram as well as the basin of attraction for the logistic map is presented for the whole range of the control parameter  $a$ , namely  $-2 \leq a \leq 4$  where the system remains finite. Equivalence of the newly found bifurcation branch to the conventional branch is shown.

**Key words:** Logistic map, bifurcation diagram, basin of attraction, equivalent transformation.

## 1. Introduction

Probably, the most thoroughly studied single system that shows bifurcations and chaos must be the logistic map which is defined as

$$x_{n+1} = a x_n (1 - x_n), \quad (1)$$

where  $n$  counts the number of iterations or the time step. Conventionally, the valid range for the variable of interest  $x$  is restricted within the range

$$0 \leq x \leq 1, \quad (2)$$

and that for the control parameter  $a$  within

$$0 \leq a \leq 4 \quad (3)$$

or sometimes within

$$1 \leq a \leq 4. \quad (4)$$

For  $a > 4$ ,  $x_n$  goes to  $-\infty$  as  $n$  increases. Thus the upper bound for  $a$  is set at  $a = 4$  unambiguously in order to maintain the system non-diverging. On the other hand, the lower bound for  $a$  becomes somewhat ambiguous, as one sees from (3) and (4). Since the fixed point at  $x = 0$  is stable for the region  $-1 \leq a \leq 1$ , the criterion to determine the lower bound for  $a$  cannot be the divergence in  $x$ . It usually depends on the interest of the researcher who deals with the map (1) which range, (3) or (4), should be taken.

In this paper we show that the logistic map (1) has another bifurcation behavior for negative  $a$  values, and for  $a < -2$  divergence of  $x$  takes place; hence the complete bifurcation diagram can be drawn without

the ambiguity mentioned above. Although the condition (2) is violated in the newly found bifurcation behaviors,  $x$  is still bounded. There are various equivalents to (1), obtained by suitable variable transformations. It will also be shown that only the type of (1) has this bidirectional bifurcation diagram. Furthermore, the equivalent transformations that transform the newly found bifurcation branch for  $1 \geq a \geq -2$  into the well-known branch for  $1 \leq a \leq 4$  are given. From the observation of the basin of attraction for the whole new range  $-2 \leq a \leq 4$  it is concluded that (4) is appropriate to employ when one deals with the familiar positive side of the bifurcation behaviors of (1). In this regard, interpretation of the behavior of (1) for  $0 < a < 1$  as extinction in the application of (1) to population dynamics is criticized. The fact that the fixed point at  $x = 0$  is stable for  $-1 < a < 1$  is well-known, and even some textbooks mention it, for example [1]. Also the fact that the map (1) becomes divergent for  $a < -2$  is already known. (See, e.g., [2]). But as far as the present authors know the existence of the period-doubling bifurcation and chaos for  $-2 \leq a \leq -1$  has never been reported.

In the next section the complete bifurcation diagram for (1) is presented. In Sect. 3 the basin of attraction for the whole new range of  $a$  is shown, and the interpretation of the attractor for  $0 < a < 1$  when the map (1) is applied to population dynamics is critically discussed. In Sect. 4 absence of the newly found bifurcation behavior in two frequently used equivalent maps to (1) is shown. In the final section it is shown that the bifurcation behavior for  $1 \geq a \geq -2$  is equivalent to that for  $1 \leq a \leq 4$ . There also a comment is given.

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## 2. Bifurcation Diagram

Today almost all the textbooks on chaos carry the bifurcation diagram for the logistic map (1) for the parameter range (3) or (4), so one is apt to take it for granted. However, it should be mentioned every time one deals with the diagram that the first systematic analysis of the bifurcation diagram for (1) was made by Grossmann and Thomae in this journal [3]. Now we extend the scope.

The complete bifurcation diagram for the logistic map (1) obtained numerically is shown in Figure 1. The fixed point or 1-cycle at  $x=0$  is stable for  $-1 < a < 1$ , where the fixed point at  $x=1-1/a$  is unstable. The stable one at  $x=0$  becomes unstable at  $a=-1$ , where the period 2 attractor or 2-cycle, which is given as

$$x_{\pm} = \frac{1}{2a}(a+1 \pm \sqrt{(a-3)(a+1)}), \quad (5)$$

is born. The expression for the 2-cycle (5) is, of course, the same as that for the case of positive  $a$ ; note, however, that  $x_{\pm}$  becomes real again for  $a < -1$ . The stability condition for (5) is fulfilled for

$$3 < a < 1 + \sqrt{6} \quad (6)$$

and

$$1 - \sqrt{6} < a < -1. \quad (7)$$

For the region  $a < 1 - \sqrt{6}$ , like for  $a > 1 + \sqrt{6}$ , we must rely solely on numerical calculation. The period-doubling bifurcation is observed also for the negative side leading to chaos. The fully developed chaos is seen for  $a = -2$ , where  $x_n$  is bounded in the range  $[-0.5, 1.5]$ . For  $a < -2$ ,  $x_n$  goes to  $\infty$  as  $n$  increases, hence the lower bound for  $a$  can be set to be at  $a = -2$  according

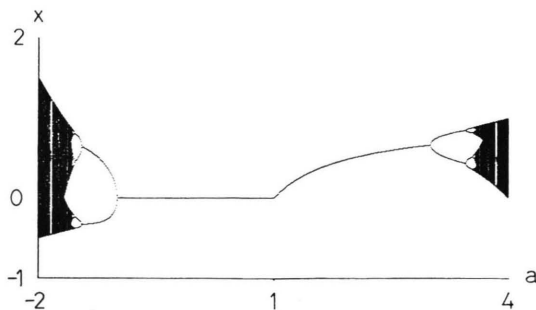


Fig. 1. The complete bifurcation diagram for the logistic map (1). The attractors corresponding to negative  $a$  values are newly found.

to the same criterion that determines the upper bound for  $a$  to be at  $a=4$ . This is the complete bifurcation diagram for (1).

## 3. Basin of Attraction

Now, that we have the complete bifurcation diagram for the logistic map (1), we next consider the basin of attraction.

For  $a > 1$  an orbit starting from any point in  $x > 1$  and  $x < 0$  goes to  $-\infty$ , whereas an orbit from any point in  $0 \leq x \leq 1$  stays within the region  $[0, 1]$ . Therefore, the boundary separating the initial points that produce the bounded and the diverging orbits are  $x=0$  and  $x=1$  for  $1 \leq a \leq 4$ .

On the other hand, for  $0 < a < 1$  any  $x$  that satisfies

$$1 - \frac{1}{a} < x < \frac{1}{a} \quad (8)$$

is attracted to the fixed point  $x=0$ . This fact is already mentioned in Holmgren's textbook [4]. For negative  $a$  values, the boundary that separates the bounded and diverging areas are also  $x=1-1/a$  and  $x=1/a$ ; for  $-1 < a < 0$  an orbit starting from any point in the region

$$\frac{1}{a} < x < 1 - \frac{1}{a} \quad (9)$$

is attracted to the fixed point  $x=0$ , whereas for  $-2 < a < -1$  the initial points in the range (9) are attracted to corresponding attractors shown in Fig. 1 and never go to infinity. The boundaries between the converging and diverging points are depicted in Fig. 2 for the whole range of the newly found valid region of  $a$  for the map (1).

One of the reasons why the map (1) has become the most famous nonlinear system must be that May introduced the system as an example of bifurcation and chaos in the context of population dynamics of a certain kind of insects [5]. The interpretation was so appealing that many textbooks explain the behaviors of (1) in terms of population dynamics ever since. There,  $x_n$  in (1) is interpreted as the relative population  $x_n = N_n/N_{\max}$ , where  $N_n$  is the actual population of the insects of interest for the  $n$ -th year and  $N_{\max}$  is the possible upper limit of  $N_n$ . Therefore  $x_n$  must be in the range  $0 \leq x_n \leq 1$  for any number of years, i.e.,  $n=0, 1, 2, \dots$ . The fact that the relative population set-

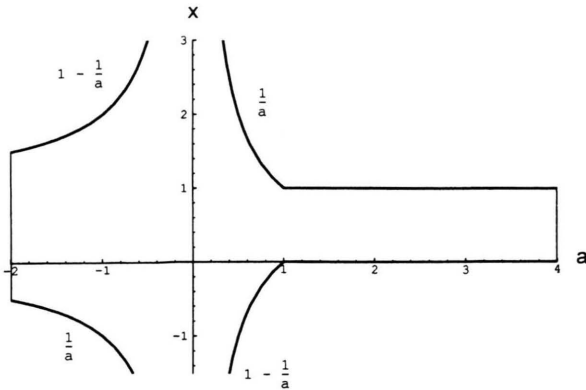


Fig. 2. The basin boundaries for the attractors shown in Fig. 1. The range of the basin of attraction for  $-2 \leq a < 1$  is completely different from that for  $1 < a \leq 4$  which has been widely studied.

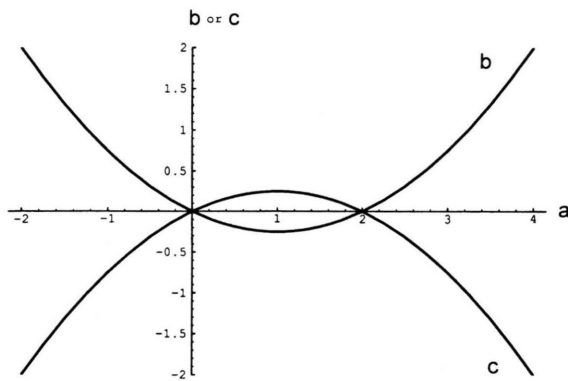


Fig. 3. The relations between the control parameters  $b$  in the map (10) and  $c$  in (12) to  $a$  in (1). The regions  $-2 \leq a < 1$  and  $1 < a \leq 4$  are degenerated in the  $b$ - or  $c$ -space.

ties in at the fixed point  $x=0$  for  $0 < a < 1$  is frequently interpreted as extinction of the insects. (For example see the textbook by Strogatz [6].) However, we see from Fig. 2 that the basin of attraction for the region  $0 < a < 1$  extends over both limits  $x=0$  and  $x=1$ , which means that there the basin of attraction is different from that for the region  $1 \leq a \leq 4$ . The final state of  $x$  for  $0 < a < 1$  apparently represents extinction, but the evolution equation (1) itself is not restricted within the same range as that for the region  $1 \leq a \leq 4$ ; in other words,  $x_n$  can exceed 1 or can be negative during the evolution. This shows that the map (1), when applied to population dynamics, works only for the region (4) and cannot properly describe extinction.

#### 4. Equivalent Maps

It is well-known that the map (1) can be transformed into different forms which show essentially the same bifurcation behaviors. The most frequently treated expressions might be the following two; one is

$$y_{n+1} = 1 - b y_n^2, \quad (10)$$

which is obtained from (1) by a set of transformations

$$y_n = \frac{a}{b} \left( x_n - \frac{1}{2} \right), \quad b = \frac{a(a-2)}{4}, \quad (11)$$

and the other is

$$z_{n+1} = z_n^2 + c, \quad (12)$$

derived by

$$z_n = a \left( \frac{1}{2} - x_n \right), \quad c = \frac{a(2-a)}{4} \quad (13)$$

from (1). It mostly depends on the tastes of the authors which of the three expressions (1), (10) or (12) is employed because as far as the period-doubling bifurcation leading to chaos is concerned, (10) and (12) show essentially the same behaviors as (1). Hao uses (10) [7] and Devaney employs (12) [8] in their textbooks, respectively. The range  $1 < a < 4$  in (1) corresponds to the range  $-\frac{1}{4} < b < 2$  in (10) and  $\frac{1}{4} > c > -2$  in (12), as is readily seen from Fig. 3 where  $b$  and  $c$  are depicted as functions of  $a$ .

The fixed point or 1-cycle for (10) is given as

$$y = \frac{1}{2b} (-1 + \sqrt{1+4b}), \quad (14)$$

and that for (12) as

$$z = \frac{1}{2} (1 - \sqrt{1-4c}). \quad (15)$$

For  $b < -\frac{1}{4}$  and  $c > \frac{1}{4}$ , the above expressions become imaginary, hence the corresponding bifurcation diagrams abruptly terminate at  $b = -\frac{1}{4}$  for the map (10) and at  $c = \frac{1}{4}$  for (12). Actually, orbits for  $b < -\frac{1}{4}$  and  $c > \frac{1}{4}$  diverge. Therefore, a bidirectional bifurcation diagram cannot be produced by the map of the form (10) or (12). We see from Fig. 3 that the ranges  $-2 \leq a \leq 1$  and  $1 < a < 4$  degenerate in  $b$  or  $c$ , hence only one bifurcation diagram emerges in  $b$  or  $c$  space.

## 5. Equivalence of Two Branches

In the last section we confirmed that a bidirectional bifurcation diagram never appears in the equivalent map (10) or (11) where only one branch of the bifurcation diagram is seen. This implies that the newly found negative-side branch of the bifurcation diagram for (1) is equivalent to the well-known positive-side branch of the diagram because there are no restrictions on the range of the parameter  $a$  in the transformations (11) and (13). In this section we give explicit expressions of the equivalent transformations from the positive-side branch to the negative-side one or vice versa.

In order to express the transformations clearly, we write the map (1) for the positive-side branch as

$$u_{n+1} = r u_n (1 - u_n), \quad (1 \leq r \leq 4), \quad (0 \leq u \leq 1) \quad (16)$$

and for the negative-side branch as

$$x_{n+1} = a x_n (1 - x_n), \quad (1 \geq a \geq -2), \quad (17)$$

where the ranges of  $x$  are  $1/a \leq x \leq 1 - 1/a$  for  $0 \geq a \geq -2$  and  $1 - 1/a \leq x \leq 1/a$  for  $1 \geq a \geq 0$ , as shown in Figure 2. If one finds a certain set of transformations that transforms (16) into (17) or vice versa, then the equivalence of the two branches is proved.

If we substitute a set of the transformations

$$u_n = \frac{a}{2-a} x_n + \frac{1-a}{2-a}, \quad r = 2-a \quad (18)$$

into (16), then we immediately have (17). Solving the transformations (18) for  $x_n$  and  $a$  and substituting them into (17), one, of course, obtains (16). Hence the newly found branch of the bifurcation diagram for (1) is equivalent to the well-known branch of the diagram for (1).

Finally we give a somewhat pessimistic comment on the new findings. At the end of Sect. 3 we pointed out that it is appropriate to restrict the parameter range to  $1 \leq a \leq 4$  when one applies the map (1) to population dynamics. When we consider applications of the whole range of the map (1), namely,  $-2 \leq a \leq 4$ , we cannot be very optimistic; there may be some physical parameters which are meaningful even when they become negative, but the fact that the valid range of the variable  $x$  for the negative-side branch differs from that for the positive-side branch seems quite unfavorable for direct physical applications. Another fact, that the attractors of the negative-side branch spread over positive and negative values at the same value of  $a$  (Fig. 1), may also make it difficult for us to apply the whole system to physical problems.

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